

Mechanics of Solids: Energetics

The motion of a continuum is governed by the two laws of thermodynamics

1. The first law is about the conservation of energy of a continuum
2. The second law is about the entropy of a continuum

Here we review these laws and examine

- their effects in the nature of constitutive equations
- introduce the potential energy for linear elastic materials

From the book: Mechanics of Continuous Media: an Introduction

1. J Botsis and M Deville, PPUR 2018.
2. J Botsis, Class Notes given during the course

Continuum mechanics review: Energetics

Energy and power involved during the motion of a continuum

Let $\omega(t)$ be the material volume of a continuous medium at time t , such that $\omega(t) \subseteq \mathcal{R}$, the deformed configuration of the body \mathcal{B} . The velocity of the motion is $\mathbf{v}(\mathbf{x}, t)$, and the material density $\rho(\mathbf{x}, t)$,

1. The kinetic energy of $\omega(t)$ is the scalar defined by the integral: $\longrightarrow E_k(t) = \int_{\omega(t)} \rho(\mathbf{x}, t) \frac{\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t)}{2} dv$ (A)

2. The internal energy of $\omega(t)$ is the scalar defined by : $\longrightarrow E_{\text{int}}(t) = \int_{\omega} \rho(\mathbf{x}, t) u(\mathbf{x}, t) dv$ (B)
Here $u(\mathbf{x}, t)$ is the internal energy per unit mass containing contributions from motion at the microscopic of molecular motion, vibrations, ...

3. The power of the volume forces given by the volume integral : $\longrightarrow \int_{\omega} \rho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) dv$ (C)

4. The power of the contact forces given by the surface integral : $\longrightarrow \int_{\partial\omega} \mathbf{t} \cdot \mathbf{v} ds = \int_{\partial\omega} \boldsymbol{\sigma} \mathbf{n} \cdot \mathbf{v} ds$
which takes the form after using the divergence theorem

$$\int_{\partial\omega} \boldsymbol{\sigma} \mathbf{n} \cdot \mathbf{v} ds = \int_{\omega} ((\mathbf{div} \boldsymbol{\sigma}) \cdot \mathbf{v} + \boldsymbol{\sigma} : \boldsymbol{\nabla} \mathbf{v}) dv \quad (\text{D})$$

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Heat transfer involved during the motion of a continuum

Another form of energy passed to the material is heat. This type of energy is in two forms:

1. Production/consumption in the body \mathcal{B} expressed as : $\longrightarrow \int_{\omega} r(\mathbf{x}, t) dv$ (E)
Here $r(\mathbf{x}, t)$ is the heat produced/received per unit time and volume. It may be due to chemical reaction, Joule effects...
2. Heat flow to the body through its boundary $\partial\omega$ given by : $\longrightarrow - \int_{\partial\omega} \mathbf{q} \cdot \mathbf{n} ds$
Here $\mathbf{q}(\mathbf{x}, t)$ is the heat flux vector and \mathbf{n} is the vector normal to the surface ds .

This integral is transformed to a volume integral which is : $\longrightarrow - \int_{\partial\omega} \mathbf{q} \cdot \mathbf{n} ds = - \int_{\omega} \operatorname{div} \mathbf{q} dv$ (F)

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FIRST PRINCIPLE OF THERMODYNAMICS: ENERGY BALANCE

The time derivative of the total energy in \mathcal{B} is equal to the sum of the power of the volume and contact forces and the rate of heat received by the material.

We combine relations (A) to (F) to express this principle as the following balance:

$$\frac{d}{dt} \int_{\omega} \rho \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2} + u \right) dv = \int_{\omega} (\rho \mathbf{b} \cdot \mathbf{v} + \operatorname{div}(\boldsymbol{\sigma} \mathbf{v}) - \operatorname{div} \mathbf{q} + r) dv$$

Using Reynolds's theorem and the principle of conservation of momentum (equations of motion), the energy balance equation (or conservation of internal energy) takes the local form:

$$\rho \frac{Du}{Dt} = \boldsymbol{\sigma} : \boldsymbol{\nabla} \mathbf{v} - \operatorname{div} \mathbf{q} + r$$

Note that the term $\boldsymbol{\sigma} : \boldsymbol{\nabla} \mathbf{v}$ can be modified as follows.
In index form we can write successively having in mind that $\sigma_{ij} = \sigma_{ji}$:

$$\sigma_{ij} \frac{\partial v_i}{\partial x_j} = \sigma_{ji} \frac{\partial v_i}{\partial x_j} = \sigma_{ij} \frac{\partial v_j}{\partial x_i} = \frac{1}{2} \sigma_{ij} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \sigma_{ij} d_{ij}$$

with

$$d_{ij} = \frac{d\varepsilon_{ij}}{dt}$$

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CONSERVATION OF MECHANICAL ENERGY

Ignoring the thermal effects the energy conservation takes the form:

$$\begin{aligned}\frac{D}{Dt} (E_k(t) + E_{\text{int}}(t)) &= \int_{\omega} \rho \frac{D}{Dt} \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) dv + \int_{\omega} \rho \frac{Du}{Dt} dv \\ &= \int_{\omega} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial\omega} \mathbf{t} \cdot \mathbf{v} ds.\end{aligned}$$

And the internal energy balance $\rho \frac{Du}{Dt} = \boldsymbol{\sigma} : \nabla \mathbf{v} - \text{div } \mathbf{q} + r$ becomes:

$$\rho \dot{u} = \rho \frac{Du}{Dt} = \boldsymbol{\sigma} : \nabla \mathbf{v} \quad \text{with} \quad \boldsymbol{\sigma} : \nabla \mathbf{v} = \boldsymbol{\sigma} : \mathbf{d}$$


$$\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{d}$$

Balance of mechanical energy:

$$\int_{\omega} \rho \frac{D}{Dt} \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) dv + \int_{\omega} \boldsymbol{\sigma} : \mathbf{d} dv = \int_{\omega} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial\omega} \mathbf{t} \cdot \mathbf{v} ds$$

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CONSERVATION OF MECHANICAL ENERGY

Balance of mechanical energy:

$$\int_{\omega} \rho \frac{D}{Dt} \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2} \right) dv + \int_{\omega} \boldsymbol{\sigma} : \mathbf{d} dv = \int_{\omega} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial\omega} \mathbf{t} \cdot \mathbf{v} ds$$

The last energy balance equation is expressed in Spatial coordinates.
Implementing the equation of motion we can express it in material coordinates.

$$\begin{aligned} \int_{\Omega} P_0 \frac{D}{Dt} \left(\frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) dV + \int_{\Omega} \mathbf{P} : \dot{\mathbf{F}} dV \\ = \int_{\Omega} P_0 \mathbf{B} \cdot \mathbf{V} dV + \int_{\partial\Omega} \mathbf{T} \cdot \mathbf{V} dS. \end{aligned}$$

An important result out of this analysis is the following power equality:

$$J \rho \dot{u} = P_0 \dot{U} = J \boldsymbol{\sigma} : \mathbf{d} = \mathbf{P} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{E}}$$

where $u(\boldsymbol{\chi}(\mathbf{X}, t), t) = U(\mathbf{X}, t)$ and J is the jacobian parameter.

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SECOND PRINCIPLE OF THERMODYNAMICS: RAMIFICATIONS OF ENTROPY

For a material volume, the material derivative of entropy is always greater than or equal to the sum of the distribution of entropy sources in the body and the entropy flux across the surface:

$$\frac{d}{dt} \int_{\omega} \rho s \, dv \geq \int_{\omega} \frac{r}{T} \, dv - \int_{\partial\omega} \frac{\mathbf{q} \cdot \mathbf{n}}{T} \, ds$$

Here s is the entropy per unit mass. By applying the Reynold's transport theorem, and accounting for the conservation of mass and the divergence theorem for the surface integral, we obtain the local form as follows:

$$\rho \frac{Ds}{Dt} \geq \frac{r}{T} - \operatorname{div} \left(\frac{\mathbf{q}}{T} \right)$$

Eliminating the term r using, $\rho \frac{Du}{Dt} = \boldsymbol{\sigma} : \boldsymbol{\nabla} \mathbf{v} - \operatorname{div} \mathbf{q} + r$

we obtain the Clausius-Duhem inequality, which must be satisfied by any thermodynamic process:

$$\rho \frac{Ds}{Dt} \geq \frac{1}{T} \left(\rho \frac{Du}{Dt} - \boldsymbol{\sigma} : \mathbf{d} \right) + \frac{1}{T^2} \mathbf{q} \cdot \boldsymbol{\nabla} T$$

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SECOND PRINCIPLE OF THERMODYNAMICS

$$\rho \frac{Ds}{Dt} \geq \frac{1}{T} \left(\rho \frac{Du}{Dt} - \boldsymbol{\sigma} : \boldsymbol{d} \right) + \frac{1}{T^2} \boldsymbol{q} \cdot \boldsymbol{\nabla} T$$

We introduce the Helmholtz specific free energy f and eliminate the term with the entropy rate,

$$f = u - Ts \Rightarrow \rho \frac{Ds}{Dt} = -\frac{\rho}{T} \frac{Df}{Dt} + \frac{\rho}{T} \frac{Du}{Dt} - \rho s \frac{DT}{Dt}$$

Introduce it in the Clausius - Duhem inequality to obtain:

$$\rho \frac{Df}{Dt} \leq \boldsymbol{\sigma} : \boldsymbol{d} - \rho s \frac{DT}{Dt} - \frac{\boldsymbol{q} \cdot \boldsymbol{\nabla} T}{T}$$

Note that the Clausius-Duhem inequality can also be easily expressed in the material description. In that case, the contact force power $\boldsymbol{\sigma} : \boldsymbol{d}$ can be expressed as $\boldsymbol{P} : \dot{\boldsymbol{F}}$ Where \boldsymbol{P} is the first Piola-Kirchhoff stress tensor and $\dot{\boldsymbol{F}}$ is the rate of the deformation gradient tensor \boldsymbol{F} .

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SECOND PRINCIPLE OF THERMODYNAMICS FOR CLASSICAL ELASTICITY

1. Consider an elastic solid subjected to small displacements and infinitesimal strains.
2. We express all quantities in terms of their material description and use lowercase symbols for convenience.
3. *We assume that geometric changes are negligible and that the deformation process takes place slowly enough so that thermodynamic equilibrium is maintained in the entire body at all times.*

In this case, the internal energy and Helmholtz free energy densities are: $\longrightarrow u = u(\boldsymbol{\varepsilon}, T) \quad f = f(\boldsymbol{\varepsilon}, T)$

We consider adiabatic process $\mathbf{q} = 0$.

Due to the assumed reversibility of all phenomena and small strains, $\mathbf{d} = \dot{\boldsymbol{\varepsilon}}$ the Clausius- Duhem inequality becomes an equality:

$$\boxed{\rho \frac{Df}{Dt} \leq \boldsymbol{\sigma} : \mathbf{d} - \rho s \frac{DT}{Dt} - \frac{\mathbf{q} \cdot \boldsymbol{\nabla} T}{T}} \longrightarrow \boxed{\rho \left(\frac{Du}{Dt} - T \frac{Ds}{Dt} \right) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = \rho \left(\frac{Df}{Dt} + s \frac{DT}{Dt} \right)}$$

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SECOND PRINCIPLE OF THERMODYNAMICS FOR CLASSICAL ELASTICITY

$$\rho \left(\frac{Du}{Dt} - T \frac{Ds}{Dt} \right) = \sigma : \dot{\epsilon} = \rho \left(\frac{Df}{Dt} + s \frac{DT}{Dt} \right)$$



$$\frac{1}{\rho} \sigma d\epsilon - s dT = df$$

$$s = -\frac{\partial f}{\partial T}$$

$$f = f(\epsilon, T) \Rightarrow df = \frac{\partial f}{\partial \epsilon_{ij}} d\epsilon_{ij} + \frac{\partial f}{\partial T} dT$$

$$\frac{1}{\rho} \sigma_{ij} = \frac{\partial f}{\partial \epsilon_{ij}}$$




Helmholtz Free Energy is a stress potential

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SECOND PRINCIPLE OF THERMODYNAMICS FOR CLASSICAL ELASTICITY

The results show that for an adiabatic process the free energy f is a potential for the stress tensor σ / ρ and entropy s :


$$\frac{1}{\rho} \sigma_{ij} = \frac{\partial f}{\partial \varepsilon_{ij}} \quad s = -\frac{\partial f}{\partial T}$$

Consider an isothermal process. In this case the free energy f is only a function only of ε_{ij} .

Expand f in the neighborhood of the natural state ($\sigma_{ij}|_0 = 0$) to obtain:


$$f = f_0 + \left. \frac{\partial f}{\partial \varepsilon_{ij}} \right|_0 \varepsilon_{ij} + \dots \quad \text{Blue arrow} \quad (f - f_0) \text{ is quadratic in } \varepsilon_{ij}$$

W : is a strain energy for adiabatic or isothermal processes.

Consider the identity

$$\frac{\partial}{\partial \varepsilon_{ij}} (\rho(f - f_0)) = \rho \frac{\partial f}{\partial \varepsilon_{ij}} + (f - f_0) \frac{\partial \rho}{\partial \varepsilon_{ij}}$$

$$W = \rho(f - f_0)$$


$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

Due to the kinematic linearization $\rightarrow \rho = \rho_0(1 + O(\varepsilon)) \approx \rho_0$


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THERMOELASTICITY

Assumption of small strains and displacements and small deviations from a reference temperature T_0 .

For an approximate theory, we expand $f(\boldsymbol{\varepsilon}, T)$ in a Taylor series in the neighborhood of $\boldsymbol{\varepsilon} = \mathbf{0}$ and $T = T_0$ until the quadratic term.

Zero stress for $\boldsymbol{\varepsilon} = \mathbf{0}$ and $T = T_0$ thus, no linear terms in $\boldsymbol{\varepsilon}$.


$$\rho f = \rho f_0 - \rho s_0(T - T_0) + \frac{\lambda}{2} \varepsilon_{ii} \varepsilon_{kk} + \mu \varepsilon_{ij} \varepsilon_{ij} + \varepsilon_{ij} c_{ij}(T - T_0) - \frac{\rho c}{2T_0} (T - T_0)^2$$

For an isotropic material

$$c_{ij} \varepsilon_{ij} (T - T_0) = -(3\lambda + 2\mu) \alpha \varepsilon_{kk} (T - T_0)$$

From the definition

$$\frac{1}{\rho} \sigma_{ij} = \frac{\partial f}{\partial \varepsilon_{ij}}$$

$$\sigma_{ij} = \rho \frac{\partial f}{\partial \varepsilon_{ij}} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - (3\lambda + 2\mu) \alpha (T - T_0) \delta_{ij}$$

or

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu} \left(\boldsymbol{\sigma} + \left(2\mu \alpha (T - T_0) - \frac{\lambda}{3\lambda + 2\mu} \text{tr } \boldsymbol{\sigma} \right) \mathbf{I} \right)$$

Terms due to temperature.

α the thermal expansion coefficient

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THERMOELASTICITY: STRESS-STRAIN-TEMPERATURE RELATIONS

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu} \left(\boldsymbol{\sigma} + \left(2\mu\alpha(T - T_0) - \frac{\lambda}{3\lambda + 2\mu} \text{tr } \boldsymbol{\sigma} \right) \mathbf{I} \right)$$

$$\varepsilon_{ij} = \frac{1}{E} \left((1 + \nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij} \right) + \alpha\delta_{ij}\Delta T$$

$$\begin{aligned}\varepsilon_{11} &= \frac{1}{E} (\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}) + \alpha\Delta T; \\ \varepsilon_{22} &= \frac{1}{E} (\sigma_{22} - \nu\sigma_{33} - \nu\sigma_{11}) + \alpha\Delta T; \\ \varepsilon_{33} &= \frac{1}{E} (\sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22}) + \alpha\Delta T; \\ \varepsilon_{12} &= \frac{1}{2G} \sigma_{12}; \quad \varepsilon_{23} = \frac{1}{2G} \sigma_{23}; \quad \varepsilon_{31} = \frac{1}{2G} \sigma_{31}\end{aligned}$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\delta_{ij}.$$

$$\begin{aligned}\sigma_{11} &= \frac{E}{(1 + \nu)(1 - 2\nu)} \left((1 - \nu)\varepsilon_{11} + \nu\varepsilon_{22} + \nu\varepsilon_{33} \right) - \frac{E}{1 - 2\nu} \alpha\Delta T; \\ \sigma_{22} &= \frac{E}{(1 + \nu)(1 - 2\nu)} \left((1 - \nu)\varepsilon_{22} + \nu\varepsilon_{33} + \nu\varepsilon_{11} \right) - \frac{E}{1 - 2\nu} \alpha\Delta T; \\ \sigma_{33} &= \frac{E}{(1 + \nu)(1 - 2\nu)} \left((1 - \nu)\varepsilon_{33} + \nu\varepsilon_{11} + \nu\varepsilon_{22} \right) - \frac{E}{1 - 2\nu} \alpha\Delta T; \\ \sigma_{12} &= \frac{E}{1 + \nu} \varepsilon_{12}; \quad \sigma_{23} = \frac{E}{1 + \nu} \varepsilon_{23}; \quad \sigma_{31} = \frac{E}{1 + \nu} \varepsilon_{31}\end{aligned}$$